HIGHER COLEMAN THEORY TALK

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1. Introduction

1.1. **Goals.** The goal of these two talks is to give an overview of the article [1]. After giving a general overview of the results under some simplifying assumptions, we will spend the rest of our time discussing the case of GSp_4.

1.2. **Singular cohomology of Shimura varieties.** Let (G, X) be a Shimura datum, let $K \subset G(\mathbb{A}_f)$ be a (sufficiently small) compact open subgroup and let

$$S_K(G, X) := G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K$$

be the corresponding Shimura variety, which is a smooth variety of dimension d over $\mathbb{C}$, let us assume that it is also compact. One generally expects that all the 'interesting' cohomology classes in the singular cohomology

$$H^*_{\text{sing}}(S_K(G, X), \mathbb{Z})$$

lie in middle degree d [1]. For example if we tensor with $\mathbb{C}$, then by Matsushima's formula there is a Hecke-equivariant bijection (for the action of the Hecke algebra of $\mathbb{C}[K\backslash G(\mathbb{A}_f)/K]$)

$$H^i_{\text{sing}}(S_K(G, X), \mathbb{Z}) \otimes \mathbb{C} \simeq \bigoplus_{\pi} (\pi^{\infty})^K \otimes H^i(\mathfrak{g}, \pi_{\infty})^{m(\pi)}$$

where π runs over automorphic representations of G that are cohomological (with respect to the trivial representation) and $H^i(\mathfrak{g}, -)$ denotes Lie algebra cohomology. In this case the 'interesting' cohomology classes are those associated to tempered automorphic representations, and it is a Theorem of Borel-Wallach that these are indeed concentrated in degree d. See the notes [6] from Joaquin's talk in the study group last winter for more details.

Another example would be the Theorem of Caraiani-Scholze [2] that

$$H^*_{\text{sing}}(S_K(G, X), \mathbb{F}_{\ell})_{\mathfrak{m}}$$

is concentrated in degree d for certain compact unitary Shimura varieties, under some assumptions on $\mathfrak{m}$. Here $\mathfrak{m}$ is a maximal ideal of the Hecke algebra; localising at $\mathfrak{m}$ means looking at the piece of cohomology with fixed mod ℓ eigenvalues determined by $\mathfrak{m}$. The condition on $\mathfrak{m}$ is that the associated Galois representation $\rho_{\mathfrak{m}}$ is generic at some prime $p \neq \ell$ (which means that the ratios of the Frob_p eigenvalues are not equal to p).

One of the main results of the Boxer-Pilloni paper is a result about

$$H^*_{\text{sing}}(S_K(G, X), \mathbb{Q}_p),$$

[1] For more general locally symmetric spaces, one expects instead a range of degrees of length ℓ_0, the so-called 'defect' of the group

or rather about

$$H^*_{\text{sing}}(K^p) := \varinjlim_{K_p} H^*_{\text{sing}}(S_{K^pK_p}(G,X),\mathbb{Q}_p),$$

where $K^p \subset G(\mathbb{A}_f^p)$ is a fixed compact open subgroup and the projective limit is indexed by compact open subgroups $K_p \subset G(\mathbb{Q}_p)$. This cohomology group has an action of $G(\mathbb{Q}_p)$, so that we can define a finite slope subspace.

Theorem 1.2.1 (Theorem 1.10 of [1])**.** *Suppose that (G,X) is of abelian type, that $S_K(G,X)$ is compact and that $G_{\mathbb{Q}_p}$ is quasi-split, then*

$$H^*_{sing}(K^p)^{ss}$$

is concentrated in middle degree d, where ss means small slope.

1.2.2. *Automorphic local systems.* Associated to an irreducible representations W_ν of G of highest weight ν, there is an automorphic local system $\mathcal{W}_\nu$ of rank equal to the dimension of W_ν. Its cohomology with $\mathbb{C}$-coefficients can also be understood in terms of automorphic forms. For example on the modular curve we get (Tate twists of) symmetric powers of the standard local system, which is the rational Tate module of the universal elliptic curve.

There is a classical vanishing theorem of Faltings, who shows that

$$H^*_{\text{sing}}(S_K(G,X),\mathcal{W}_\nu)$$

is concentrated in middle degree, as long as ν is sufficiently regular. He does this by describing the cohomology of $\mathcal{V}_\nu$ in terms of the cohomology of certain automorphic vector bundles $\mathcal{V}_\kappa$ related to $\mathcal{W}_\nu$ and then proving vanishing results for those. Actually, this is also the way Boxer and Pilloni prove Theorem 1.2.1, so we need to discuss automorphic vector bundles.

1.3. **Automorphic vector bundles.** Let $M \subset P \subset G$ be the Levi- and parabolic subgroup determined by the Shimura datum (G,X) choose $T \subset B \subset P$ be a maximal torus and a Borel. For each irreducible representation V_κ of M of highest weight κ there is an automorphic vector bundle $\mathcal{V}_\kappa$. For example for the modular curve $M = (\mathbb{G}_m)^2$ and the automorphic vector bundles are given by powers of the Lie algebra of the universal elliptic curve (the second parameter only twists the action of the Hecke-algebra).

1.3.1. *Notation.* Let $X_*(T)^+$ denote the dominant characters of the torus, and let $X^*(T)^{M,+}$ denote the M-dominant cocharacters. Note that $X^*(T)^+ \subset X^*(T)^{M,+}$, for example if M is a torus then all cocharacters are M-dominant. Let W denote the Weyl group of G, which contains the Weyl group $W_M \subset W$ and we let ${}^MW$ denote a set of coset representatives of W/W_m of minimal length. Let ρ be half the sum of the positive roots and let $w_{0,M} \subset W_M$ be the longest element.

There is a combinatorial process which associates to a dominant coweight $\nu \in X^*(T)^+$ a collection of M-dominant coweights and Faltings‘ BGG resolution describes the cohomology of the local system $\mathcal{W}_\nu^\vee$ in terms of the cohomology of the automorphic vector bundles associated to κ.

$$H^*_{\text{sing}}(S_K(G,X),\mathcal{W}_\nu^\vee) = \bigoplus_{w \in {}^MW} H^{i-d+l(w)}\left(S_K(G,X),\mathcal{V}_{-w_{0,M}(w(\nu+\rho)-\rho}\right).$$

For example in the case of the modular curve then $W = \{1,s\}$ and $W_M = \{1\}$ so that ${}^MW$ has two elements and we recover the Hodge decomposition

$$H^1(Y_1(N),\mathcal{W}_k) = H^0(\mathcal{V}_{k+2}) \oplus H^1(\mathcal{V}_{-k}).$$

1.4. **Coherent cohomology of automorphic vector bundles.** It is not true that *every* automorphic vector bundle is associated to an automorphic local system by the above process. For example modular forms of weight 1 do not contribute to the cohomology of an automorphic local system on the modular curve. At the same time, we do not expect that the 'interesting' cohomology classes of all automorphic vector bundles are concentrated in a single degree. We only expect this for automorphic vector bundles associated to 'sufficiently regular weight κ', and for those automorphic vector bundles the degree in which we expect the 'interesting' cohomology classes depends on κ!

Let $C(\kappa)^+ = \{w \in W \mid w^{-1} w_{0,M}(\kappa+\rho) \in X^*(T)^-_{\mathbb{Q}}\}$ and let $\ell_{\min}(\kappa), \ell_{\max}(\kappa)$ be the minimum respectively maximum length of an element in $C(\kappa)^+$. Then we expect that all the 'interesting' cohomology classes of

$$H^*(S_K(G,X), \mathcal{V}_\kappa)$$

are concentrated in the range $[\ell_{\min}(\kappa), \ell_{\max}(\kappa)]$. For example on the modular curve we expect 'interesting' cohomology classes in a single degree unless $\kappa = 1$, when we expect 'interesting' cohomology classes in both degrees. In general, this range of degrees is a single degree if $\kappa + \rho$ is Weyl-conjugate to a unique anti-dominant element (that is, if $\kappa + \rho$ is regular). As before we define

$$H^*(K^p, \kappa) := \varinjlim_{K_p} H^*(S_{K^p K_p}(G,X), \mathcal{V}_\kappa).$$

Then another main result of Boxer-Pilloni is that

Theorem 1.4.1 (Theorem 1.5 of [1]). *Assume that (G,X) is of abelian type, that $S_K(G,X)$ is compact and that $G_{\mathbb{Q}_p}$ is quasi-split, then*

$$H^*(K^p, \kappa)^{ss^M(\kappa)},$$

is concentrated in the range of degrees $[\ell_{min}(\kappa), \ell_{max}(\kappa)]$. Here ss stands for small slope.

Remark 1.4.2. There are other vanishing results in the literature that work without the finite slope condition, but they all have to assume that κ is sufficiently regular in some way.

1.5. **Constructions and main results.** The main new construction in the paper are certain 'local' overconvergent cohomology groups associated to $w \in {}^M W$ and an automorphic vector bundle $\mathcal{V}_\kappa$. These will be defined as the cohomology of the automorphic vector bundle on certain open subsets of the adic generic fibre (we are dropping K^p from the notation now)

$$\mathcal{S}_{K_p},$$

with certain support conditions. Basically we will construct a filtration by closed subsets

$$\mathcal{S}_{K_p} = Z_0 \supset Z_1 \supset \cdots \supset Z_d \supset Z_{d+1} = \emptyset$$

and study the E_1 spectral sequence associated to this filtration. The objects of the E_1 page will be the cohomology groups

$$R\Gamma_d(V_\kappa) := R\Gamma_{Z_d \setminus Z_{d+1}}(\mathcal{S}_{K_p} \setminus Z_{d+1}, \mathcal{V}_\kappa)$$

and in fact

$$R\Gamma_d = \bigoplus_{\substack{w \in {}^M W \\ l(w) = d}} R\Gamma_w(\mathcal{V}_\kappa).$$

Our filtration of the Shimura variety is pulled back from a filtration of the adic flag variety $\mathcal{FL}$ via the Hodge-Tate period map

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\pi_{\mathrm{HT}}} & \mathcal{FL} \\ \downarrow & & \downarrow \\ \mathcal{S}_{K_p} & \longrightarrow & K_p\backslash\mathcal{FL}. \end{array}$$

The filtration on the flag variety comes from the Bruhat-stratification on the special fiber. The shape of the spectral sequence is a 'rhombus' and its diagonal is given by the Cousin complex

$$H^0_{Z_0/Z_1}(\mathcal{S}_{K_p}, \mathcal{V}_\kappa) \to H^1_{Z_1/Z_2}(\mathcal{S}_{K_p}, \mathcal{V}_\kappa) \to \cdots \to H^d_{Z_{d-1}/Z_d}(\mathcal{S}_{K_p}, \mathcal{V}_\kappa).$$

The whole spectral sequence has an action of the Hecke-algebra $\mathbb{Q}_p[K_p\backslash G(\mathbb{Q}_p)/K_p]$ and in fact it acts by compact operators on all the terms on the E_1 page. This means that we can pass to finite slope subspaces.

Theorem 1.5.1 (Theorem 5.18 of [1])**.** *The top half of the rhombus (everything strictly above the diagonal) is zero. To be precise, the complexes*

$$R\Gamma_w(\mathcal{V}_\kappa)^{fs}$$

are concentrated in degrees $[0, \ell(w)]$.

Conjecture 1.5.2 (Conjecture 5.20 of [1])**.** *The bottom half of the rhombus (everything strictly below the diagonal) is zero, in other words, the complex*

$$R\Gamma_w(\mathcal{V}_\kappa)^{fs}$$

is concentrated in degrees $[\ell(w), d]$. *Therefore the cohomology* $H^i(\mathcal{S}_{K_p}, \mathcal{V}_\kappa)$ *is computed by the Cousin complex).*

Remark 1.5.3. There is a natural pairing between the top half and the bottom half of the spectral sequence. If we knew it was perfect, then the vanishing of the top half would give vanishing of the bottom half. We do know that the perfect pairing induces Serre duality on the abutment of the spectral sequence and this allows us to deduce that the cohomology groups $H^i(\mathcal{S}_{K_p}, \mathcal{V}_\kappa)^{fs}$ are subquotients of the cohomology of the Cousin spectral sequence.

To prove these vanishing conjectures, we need to prove a lower bound on the slopes of our overconvergent cohomology groups. Then we define small slope and strictly small slope conditions such that the following result holds:

Theorem 1.5.4 (Corollary 5.65)**.** *The cohomology groups*

$$R\Gamma_w(\mathcal{V}_\kappa)^{sss}$$

are zero unless $w \in C(\kappa)^+$, *where sss stands for strictly small slope.*

Remark 1.5.5. Boxer and Pilloni conjecture stronger slope estimates than they can prove (Theorem 5.33 vs Conjecture 5.29). I think Conjecture 5.29 would imply the above theorem with sss replaced by ss (but I haven't checked this!). They mention that proving Conjecture 5.29 would require a study of integral models of these Hecke correspondences, and also that it is compatible with Conjecture 4.5 of [3]

If $\kappa+\rho$ is regular then the theorem tells us that the spectral sequence only has a single column, and it follows that it must degenerate at E_1. Then we deduce from our duality argument that there is only one nonzero term in the column, which gives the following result (a classicality theorem)

Theorem 1.5.6 (Theorem 5.66 of [1])**.** *Suppose that $\kappa+\rho$ is regular, then*

$$R\Gamma_w(\mathcal{V}_\kappa)^{sss} \simeq R\Gamma(K^p,\kappa)^{sss}$$

and it is concentrated in degree $\ell(w)$.

For more general weights κ we get a vanishing result, but not a classicality theorem (we don't expect the spectral sequence to degenerate at E_1, so we probably don't expect a classicality theorem?)

Theorem 1.5.7 (Theorem 5.69 of [1])**.** *Now let κ be arbitrary, then*

$$H^*(K^p,\kappa)^{sss}$$

is concentrated in degrees $[\ell_-(w),\ell_+(w)]$.

Remark 1.5.8. This is weaker than Theorem 1.5 of [1] which we stated at the beginning, which had ss instead of sss. The way to improve the slope bounds is to p-adically interpolate over the eigenvariety, and then prove the result for the regular weights (which are dense). To prove these results in regular weight, we relate the cohomology of automorphic vector bundles of regular weight to the cohomology of automorphic local systems, and then run an argument of Vincent Lafforgue [4] there. I haven't understood Section 6 yet, and I will probably not cover it next week.

1.5.9. *Omissions.*

- Interpolating the coefficients sheaves $\mathcal{V}_\kappa$ and eigenvarieties, applications to local-global compatibility (all of Section 6)
- Duality and the dual support conditions, small slope conditions etc.
- De Rham and rigid cohomology.
- Non-compact Shimura varieties.
- Shimura varieties of abelian type.

2. Siegel threefolds

From now on we will assume that $G_{\mathbb{Q}_p}=\mathrm{GSp}_4$ and give a detailed overview of the constructions in the Boxer-Pilloni paper. First we define GSp_4 and $T\subset B\subset P$ and the Weyl groups, root system etc carefully. We've basically copied this from Chapter 2 of [5].

Consider $V=\mathbb{Z}^{\oplus 4}$ with basis $\langle x_2,x_1,y_1,y_2\rangle$ and symplectic form Ψ given by $\langle x_i,y_i\rangle=1$. We define the group GSp_4 as the group scheme over $\mathbb{Z}$ defined by the functor sending a commutative ring R to automorphism of $V\otimes_{\mathbb{Z}} R$ preserving Ψ_R up to a scalar in R. The parabolic P is the stabiliser of the standard Lagrangian subspace $\langle y_1,y_2\rangle$ and its Levi M is isomorphic to $\mathrm{GL}_2\times\mathbb{G}_m$. We then choose a Borel B such that its intersection with M is isomorphic to the upper triangular matrices in $\mathrm{GL}_2\times\mathbb{G}_m$. We take the maximal torus $T \subset B$ isomorphic to $\mathbb{G}_m^3$ with $(t_1, t_2, c) \mapsto \operatorname{diag}(t_1c, t_2c, t_2^{-1}c, t_1^{-1}c) = t(t_1, t_2, c)$. We identify the character group with

$$\{(k_1, k_2; k) \in \mathbb{Z}^3 \mid k_1 + k_2 = k \mod 2\}$$

such that

$$(k_1, k_2; k)(t_1, t_2, c) = t_1^{k_1} \cdot t_2^{k_2} \cdot c^k$$

With our choice of Borel B the dominant characters are given by

$$X^*(T)^+ = \{(k_1, k_2; k) \in X_*(T) \mid 0 \geq k_1 \geq k_2\},$$

the positive simple roots are given by $\alpha_1 = (1, -1, 0)$ and $\alpha_2 = (0, 2, 0)$. The M-dominant cocharacters are given by

$$X^*(T)^{M,+} = \{(k_1, k_2; k) \in X_*(T) \mid k_1 \geq k_2\}$$

and the half sum of the positive roots is given by

$$\rho = (-1, -2; 0).$$

The Weyl group is generated by the reflections $s_0(k_1, k_2; k) = (k_2, k_1; k)$ and $s_1(k_1; k_2; k) = (-k_1; k_2; k)$ and the Weyl group of M is generated by s_0, such that

$$^M W = \{1, s_0, s_1 s_0, s_1 s_0 s_1\}.$$

Here is a picture of the root system

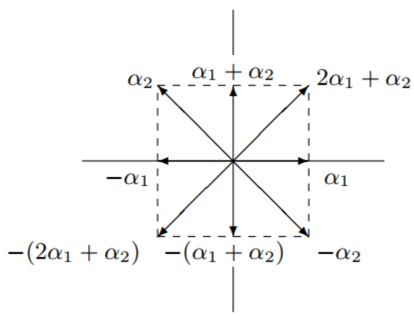

Figure 1. *The root system of* GSp_4

2.1. **Flag varieties.** The flag variety $\mathrm{FL} = P \backslash G$ is a smooth projective scheme of relative dimension 3 over $\mathbb{Z}_p$ also known as the 'Lagrangian Grassmannian' because it is the moduli spaces of Lagrangian subspaces of $\mathbb{Z}_p^{\oplus 4}$. Our fixed choice of Borel B gives us a stratification

$$G/B = \bigcup_{w \in {}^M W} C_w = \bigcup_{w \in {}^M W} (B \cap w^{-1} P w) \backslash B,$$

where C_w are smooth Bruhat cells of (relative) dimension $\ell(w)$. These also have moduli interpretations as Lagrangian subspaces $H \subset \mathbb{Z}_p^{\oplus 4}$ with certain intersection behaviour with respect to the standard flag $\langle x_1 \rangle \subset \langle x_1, x_2 \rangle$. Explicitly: The smallest Bruhat cell (a point), is the subspace where $H = \langle x_1, x_2 \rangle$, the largest Bruhat cell is the subspace where $H \cap \langle x_1, x_2 \rangle = 0$ and the middle ones are where $H \cap \langle x_1, x_2 \rangle = \langle x_1 \rangle$ or where $H \cap \langle x_1, x_2 \rangle$ is one-dimensional but not equal to $\langle x_1 \rangle$. In our case the Bruhat cells are linearly ordered.

The closure of the Bruhat cell C_w is the Schubert variety

$$X_w = \bigcup_{w' \leq w} C_w,$$

we also introduce the open subspace

$$Y_w = \bigcup_{w' \geq w} C_w.$$

Note that C_w in canonically isomorphic to an affine space of that dimension of dimension $\ell(w)$ (it is isomorphic to a product of root subgroups of G, see Lemma 3.3 of [1])

2.2. **Analytic neighbourhoods.** The promised filtration by closed subspaces on the adic generic fibre $\mathcal{FL}$ of the flag variety is defined as follows

$$Z_r = \overline{]\,\mathrm{FL}_{\geq r}[}$$

where

$$\mathrm{FL}_{\geq r} = \bigcup_{\substack{w \in {}^M W \\ \ell(w) \geq r}} C_w.$$

In our case this specialises to $\mathrm{FL}_{\geq r} = Y_w = Y_r$ where w is the unique element of length r for $0 \leq r \leq 3$.

Lemma 2.2.1. *The complement of* $\overline{]\,\mathrm{FL}_{\geq r+1}[}$ *in* $\mathcal{FL}$ *is equal to* $]\,\mathrm{FL}_{\leq r}[$ *and the intersection*

$$\overline{]Y_w}[\cap]X_w[=]C_w[_{0,\bar{0}}$$

is a partial compactification of the tube of C_w.

Remark 2.2.2. In the extreme case where w is of length zero, then

$$Z_0 = \mathcal{FL}$$

and $Z_0 \setminus Z_1$ is a partial closure of the tube of the point C_w in the special fiber of k. The cohomology with support that we are taking is just going to be the cohomology of $Z_0 \setminus Z_1$, and this is going to be concentrated in degree 0 because $Z_0 \setminus Z_1$ is acyclic. [And a similar result will hold after pulling back via the Hodge-Tate period map, more on this next week!]

3. Part II

I changed my conventions for the root system of GSp_4 compared to last week, to make sure that they agree with Boxer-Pilloni and also with Toby's talk next week. Today we are going to go construct the overconvergent cohomology groups that I mentioned existed last week, with emphasis on the case of $G = \mathrm{GSp}_4$. First, we need to discuss some basics about cohomology with support in a closed subscheme, and the action of finite flat correspondences on such cohomology groups.

3.1. Analytic continuation for modular forms.

When constructing spaces of overconvergent modular forms of weight k, one runs into the following kind of analytic continuation result. Let U be an ϵ-neighbourhood of the ordinary locus and let $T = U_p$ be the Hecke correspondence. Then $T(U) \subset U$ is a strict inclusion and so we get a map

$$T : H^0(U, V_k) \to H^0(T(U), V_k)$$

and if we compose with the natural restriction map we have defined an endomorphism of $H^0(U, V_k)$. Moreover, we can conclude that

$$H^0(U, V_k)^{fs} \simeq H^0(T(U), V_k)^{fs}$$

is an isomorphism. Here fs means finite slope and corresponds to taking the part of the cohomology where T acts with nonzero eigenvalues. The claimed isomorphism follows from the diagram

$$\begin{tikzcd}
H^0(U, V_k) \arrow[r] \arrow[d, "T"] & H^0(T(U), V_k) \arrow[d, "T"] \arrow[dl, dotted, "T"] \\
H^0(U, V_k) \arrow[r] & H^0(T(U), V_k).
\end{tikzcd}$$

Indeed, we can show that any eigenvector with nonzero T eigenvalue on the left hand side must map nontrivially to the right hand side, and similarly every such eigenvector on the right hand side must come from the left hand side. When we are doing higher Coleman theory, we will (in general) not be able to find ϵ-neighbourhoods U of ordinary loci (or Igusa varieties) such that $T(U) \subset U$, and instead we will have to work with cohomology with support. [George Boxer draws really nice pictures in talks explaining this, see for example `https://www.youtube.com/watch?v=gk1C-eR9i3A` about 20 minutes in.] We will be able to find some U such that $T(U) \subset U$ but that U will be 'too big', at least if we interpret the pictures literally.

3.2. Cohomology with support.

Let X be an adic space locally of finite type over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X. If Z is a closed subset of X then we can take cohomology with support

$$R\Gamma_Z(X, V)$$

which is the derived functor of sections with supports. This has the following properties (see Section 2.1 of [1])

- If $Z \subset Z'$ then there is a corestriction map map $R\Gamma_Z(X, V) \to R\Gamma_{Z'}(X, V)$.
- If $Z \subset X' \subset X$ for some open X', then there is a restriction map $R\Gamma_Z(X, V) \to R\Gamma_Z(X', V)$ (which is a quasi-isomorphism).
- If $f : X \to Y$ is a morphism such that $f^{-1}(Z_Y) \subset Z_X$, then there is a pullback map

$$R\Gamma_{Z_Y}(Y, \mathcal{F}) \to R\Gamma_{Z_X}(X, f^*\mathcal{F}),$$

- If $f : X \to Y$ is finite flat such that $f(Z_X) \subset Z_Y$ then there is a trace map
$$R\Gamma_Z(X, f^*\mathcal{F}) \to R\Gamma_{Z_Y}(Y, \mathcal{F}).$$

3.3. **Finite flat correspondences.** Here we follow Section 5.1 of [1]. Consider a diagram

$$\begin{tikzcd} & C \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

in the category of adic spaces locally of finite type over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. We let T and T^t denote the functions $p_2(p_1^{-1}))$ and $p_1(p_2^{-1}))$ on subsets of X.

Lemma 3.3.1. *Suppose that p_1 and p_2 are finite flat, then T and T^t take open sets to open sets and closed sets to closed sets*

Proof. Finite flat maps of adic spaces are open and closed. $\square$

From now on we will assume that p_1 and p_2 are finite flat. Suppose that we have a coherent sheaf V on X together with a map $T : p_2^*V \to p_1^*V$, then there is a map
$$T : R\Gamma_{Z\cap T(U)}(T(U), V) \to R\Gamma_{T^t(Z)\cap U}(U, V)$$
constructed as follows (we use that $p_2^{-1}(T(U)) \supset p_1^{-1}(U)$ and that $p_1^{-1}(T^t(Z)) \subset p_2^{-1}(Z)$

$$\begin{tikzcd} R\Gamma_{Z\cap T(U)}(T(U), V) \arrow[d, "p_2^*"] & R\Gamma_{T^t(Z)\cap U}(U, V) \\ R\Gamma_{p_2^{-1}(Z)\cap p_2^{-1}(T(U))}(p_2^{-1}(T(U)), p_2^*V) \arrow[d] & R\Gamma_{p_1^{-1}(T^t(Z))\cap p_1^{-1}(U)}(p_1^{-1}(U), p_1^*V) \arrow[u, "\mathrm{Tr}\, p_1"'] \\ R\Gamma_{p_2^{-1}(Z)\cap p_1^{-1}(U)}(p_1^{-1}(U), p_2^*V) \arrow[r, "T"] & R\Gamma_{p_2^{-1}(Z)\cap p_1^{-1}(U)}(p_1^{-1}(U), p_1^*V) \arrow[u] \end{tikzcd}$$

3.4. **Analytic continuation.** Suppose that we are in the same setting as the previous section, and assume that $T(U) \subset U$ and $T^t(Z) \subset Z$. Then we can consider the following diagram

$$\begin{tikzcd} & R\Gamma_{Z\cap U}(U, V) \arrow[dl] & \\ R\Gamma_{Z\cap T(U)}(T(U), V) \arrow[rr, "T"] & & R\Gamma_{T^t(Z)\cap U}(U, V) \arrow[ul] \arrow[dl] \\ & R\Gamma_{T^t(Z)\cap T(U)}(T(U), V) \arrow[ul] & \end{tikzcd}$$

We can now define endomorphisms labelled T for all the spaces in the diagram, and then we have the following proposition

Proposition 3.4.1. *Assume that T is a compact operator, then all the maps in the above diagram induce isomorphisms on finite slope spaces.*

Proof. This follows as above. □

We will often be in the situation that $T^n(U) \subset U$ and $(T^t)^n Z \subset Z$, which means that we can shrink our Z and U 'as much as we want'.

3.5. **The Hodge-Tate period map.** Our Shimura varieties of Hodge type come equipped with a Hodge-Tate period map. When $G = \mathrm{GSp}_4$, the Shimura variety $\mathcal{S}_{K^p}$ with infinite level at p 'parametrises' principally polarized abelian surfaces together with a choice of symplectic similitude

$$T_pA \simeq \mathbb{Z}_p^{\oplus 4}.$$

This is naturally a pro-étale $\mathrm{GSp}_4(\mathbb{Z}_p)$ torsor over the Shimura variety with hyperspecial level at p.

Given an abelian variety over $\mathbb{C}_p$ with a choice of basis for its Tate module, we get a point in a flag variety as follows: The Hodge-Tate filtration

$$0 \to \mathrm{Lie}(A) \to T_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p \to \omega_{A^t} \to 0$$

gives us a Lagrangian subspace $\mathrm{Lie}(A)$ of $\mathbb{C}_p^{\oplus 4} \simeq T_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p$, in other words, an element of the Lagrangian Grassmannian. Of course we need to make this work in families, and for that we need Scholze's p-adic Hodge theory paper. This works exactly the same for $G = \mathrm{GSp}_{2g}$ and can be refined to Shimura varieties of Hodge type (see [2]). [In fact there is work of David Hansen refining it to all Shimura varieties]. Let us summarise this section by the following Theorem:

Theorem 3.5.1 (Scholze, Caraiani-Scholze)**.** *There is a perfectoid space $\mathcal{S}_{K^p}$ that is the inverse limit of $\mathcal{S}_{K^pK_p}$ (but not quite in the category of adic spaces). It has an action of $G(\mathbb{Q}_p)$ and an equivariant morphism of adic spaces*

$$\pi_{HT} : \mathcal{S}_{K^p} \to \mathcal{FL}.$$

3.6. **The Hodge-Tate period map at finite level.** Boxer and Pilloni don't work with infinite level Shimura varieties, but still want to make use of the Hodge-Tate period map. The following Theorem tells us that this is possible and moreover it tells us that the Hodge-Tate period map is affine. We define a topological space $\mathcal{FL}/K_p$ using the quotient topology and we call an open subset of it affinoid if its inverse image in $\mathcal{FL}$ is affinoid. If $V \subset U$ are open subsets then we call V a rational subset of U if this is true after taking inverse images in $\mathcal{FL}$. Similarly, there is a notion of classical point.

Theorem 3.6.1 (Theorem 4.66 of [1])**.** *There is a continuous map*

$$\pi_{HT,K_p} : \mathcal{S}_{K^pK_p} \to \mathcal{FL}/K_p$$

which is equivariant for the action of the Hecke algebra at p via Hecke correspondences. Moreover, every point $x \in \mathcal{FL}/K_p$ that comes from a classical point upstairs has an affinoid neighbourhood U such that the inverse image under π_{HT,K_p} of any rational subset $V \subset U$ is affinoid.

3.6.2. *Automorphic sheaves.* Let me try to summarise 10 pages of [1] in one sentence: Hecke correspondences are cohomological correspondences for the automorphic sheaves $\mathcal{V}_k$, in other words, there are maps $p_1^*\mathcal{V}_k \to p_2^*\mathcal{V}_k$. Moreover, we can describe these maps 'explicitly' in local coordinates, this will be important for proving slope bounds later.

3.7. **Dynamics of the torus action.** Let $v : \mathbb{Q}_p \to \mathbb{R} \cup \infty$ denote the p-adic valuation normalised by $v(p) = 1$. Define

$$T^+(\mathbb{Q}_p) = \{t \in T(\mathbb{Q}_p) \mid v(\alpha(t)) \geq 0 \text{ for } \alpha \in \Phi^+\}$$

and define T^{++} to be the same except with $v(\alpha(t)) > 0$. We also want to define T^- and T^{--} where we replace $\geq$ with $\leq$ and $<$, respectively.

Lemma 3.7.1 (Lemma 3.23 of [1])**.** *If $t \in T^+(\mathbb{Q}_p)$, then*

$$]X_w[\cdot t \subset]X_w[$$

and if $t \in T^-(\mathbb{Q}_p)$ then

$$]Y_w[\cdot t \subset]Y_w[.$$

Proof. This requires an explicit description of the tube $]C_w[$ in terms of analytic root subgroups, which we haven't covered. ☐

3.8. **Dynamics of Correspondences.** Now let $K_p = \mathcal{I}$ be the Iwahori subgroup of $\mathrm{GSp}_4(\mathbb{Q}_p)$, i.e., the subgroup of $\mathrm{GSp}_4(\mathbb{Z}_p)$ whose mod p reduction lies in the Borel subgroup. Then the Hecke algebra $\mathcal{H}_{K_p} := \mathbb{Z}[K_p \backslash G(\mathbb{Q}_p)/K_p]$ acts via finite flat correspondences on $\mathcal{FL}/K_p$.

Lemma 3.8.1. *Let $t \in T^+(\mathbb{Q}_p)$ and consider the Hecke correspondence $T = K_p t K_p$, then*

$$T^n(U) \subset T^{n-1}(U) \subset \cdots \subset U$$
$$(T^t)^n(Z) \subset (T^t)^{n-1}(Z) \subset \cdots \subset Z.$$

Proof. This follows from Lemma 3.7.1 once we notice that transposition of Hecke operators switches t with t^{-1}, and the fact that $t \in T^+$ implies that $t^{-1} \in T^-$. ☐

This means it makes sense to consider the action of the Hecke algebra

$$\mathcal{H}^+_{K_p}$$

which is generated by the double cosets $K_p t K_p$ for $t \in T^+(\mathbb{Q}_p)$ (we will also consider the ++ variant). For $G = \mathrm{GSp}_4$ and $K_p = \mathcal{I}$, the Hecke algebra $\mathcal{H}^+_{K_p}$ is generated by the following Hecke operators

$$U_2 = [K_p \operatorname{diag}(p^{-1}, p^{-1}, 1, 1) K_p]$$
$$U_1 = [K_p \operatorname{diag}(p^{-1}, p^{-2}, 1, p^{-1}) K_p]$$
$$S = [pK_p], S^{-1}.$$

So if we are going to take the finite slope part, then we are just taking that part of the cohomology where U_2 and U_1 have nonzero eigenvalues.

3.9. **Overconvergent cohomologies.** Now let $w \in {}^M W$ and define

$$U_w := \pi^{-1}_{HT,K_p}(]X[_w),$$
$$Z_w := \pi^{-1}_{HT,K_p}(]Y]_w).$$

Moreover for $\kappa \in X_*(T)^{+,M}$ we define

$$R\Gamma_w(K^pK_p,\kappa) := R\Gamma_{U_w \cap Z_w}(U_w, \mathcal{V}_\kappa).$$

Using Lemma 3.8.1 we get an action of $\mathcal{H}^+_{K_p}$ on these cohomology groups via restriction-corestriction. We following result lets us take finite slope parts:

Theorem 3.9.1 (Theorem 5.10 of [1])**.** *There is an action of the Hecke algebra $\mathcal{H}^+_{K_p}$ on $R\Gamma_w(K^pK_p,\kappa)$ such that the elements of $\mathcal{H}^{++}_{K_p}$ act via compact operators. This means that the finite slope subspace*

$$R\Gamma_w(K^pK_p,\kappa)^{fs}$$

is well defined.

3.9.2. *Change of level and support condition.* The paper also defines variants of $R\Gamma_w(K^pK_p,\kappa)^+$ for levels $K_{p,m,n}$ deeper than Iwahori level and for 'allowable' support conditions

Theorem 3.9.3 (Theorems 5.13 and 5.14 of [1])**.** *The finite slope part of our overconvergent cohomology groups do not depend on the choice of level at p or the choice of allowable support condition. From now on we will denote then as*

$$R\Gamma_w(K^p,\kappa)^{fs}.$$

3.10. **Spectral sequence and the Cousin complex.** Recall from last week that there should be a spectral sequence that computes the cohomology $R\Gamma(\mathcal{S}_{K^pK_p}, \mathcal{V}_\kappa)$ in terms of the overconvergent cohomology groups. In the case of $G = \mathrm{GSp}_4$, we've drawn the first page in Figure 2 (we will write $H^i_w(K^p,\kappa)^{fs}$ for $H^iR\Gamma_w(K^p,\kappa)^{fs}$): There is no cohomology in degrees greater than three because all our adic spaces are 3-dimensional. We define the Cousin complex

$$\mathrm{Cous}(K^p,\kappa) := H^0_1(K^p,\kappa)^{fs} \longrightarrow H^1_{s_1}(K^p,\kappa)^{fs} \longrightarrow H^2_{s_1s_2}(K^p,\kappa)^{fs} \longrightarrow H^3_{s_1s_2s_1}(K^p,\kappa)^{fs}.$$

Theorem 3.10.1 (Theorem 5.18 of [1])**.** *The top half of the rhombus (everything strictly above the diagonal) is zero. To be precise, the complexes*

$$R\Gamma_w(\mathcal{V}_\kappa)^{fs}$$

are concentrated in degrees $[0, \ell(w)]$.

Proof. We give an idea of the proof: There is a long exact sequence in cohomology coming from the following exact triangle

$$R\Gamma_{Z_w\cap U_w}(U_w,\mathcal{V}_\kappa) \to R\Gamma(Z_w \cap U_w, \mathcal{V}_\kappa) \to R\Gamma(U_w \setminus Z_w, \mathcal{V}_\kappa).$$

After choosing our support condition and level favourably, then we can arrange for $Z_w \cap U_w$ to be affinoid, so that it has no higher cohomology. Then all we have to do is show that the cohomology of $R\Gamma(U_w \setminus Z_w, \mathcal{V}_\kappa)$ is supported in degrees $[0, l(w) - 1]$.

$$
\begin{tikzcd}
H^3_1(K^p,\kappa)^{fs} & & & \\
H^2_1(K^p,\kappa)^{fs} \arrow[r] & H^3_{s_1}(K^p,\kappa)^{fs} & & \\
H^1_1(K^p,\kappa)^{fs} \arrow[r] & H^3_{s_1}(K^p,\kappa)^{fs} \arrow[r] & H^2_{s_1s_2}(K^p,\kappa)^{fs} & \\
H^0_1(K^p,\kappa)^{fs} \arrow[r] & H^1_{s_1}(K^p,\kappa)^{fs} \arrow[r] & H^2_{s_1s_2}(K^p,\kappa)^{fs} \arrow[r] & H^3_{s_1s_2s_1}(K^p,\kappa)^{fs} \\
& H^0_{s_1}(K^p,\kappa)^{fs} \arrow[r] & H^1_{s_1s_2}(K^p,\kappa)^{fs} \arrow[r] & H^2_{s_1s_2s_1}(K^p,\kappa)^{fs} \\
& & H^0_{s_1s_2}(K^p,\kappa)^{fs} \arrow[r] & H^1_{s_1s_2s_1}(K^p,\kappa)^{fs} \\
& & & H^0_{s_1s_2s_1}(K^p,\kappa)^{fs}
\end{tikzcd}
$$

Figure 2. *The first page of the spectral sequence computing the cohomology of a Siegel threefold in terms of overconvergent cohomologies. The Cousin complex is the length four complex in the middle*

On the flag variety we know that $]X_w[\setminus]Y_w[=\cup_{w'<w}]C_w[$ and it makes some sense to me that we would expect this to have cohomology in degrees $[0,l(w)-1]$ since it is basically equal to the tube of a closed subvariety of the special fiber of $\mathcal{FL}$ of dimension $l(w)-1$. However that is not how the argument in the paper goes: It once again uses explicit descriptions of our tubes in terms of analytic root subgroups and then they shrink their parameters enough so that $U_w \setminus Z_w$ can be covered by $l(w)-1$ acyclic spaces. To be honest I don't really understand what is going on. $\square$

So now we have cut down our spectral sequence to a much more manageable size (but recall that we conjecture that the bottom half is also zero, so that the spectral sequence is just 'equal' to the Cousin complex).

3.11. **Slopes.** Let us work with $G=\mathrm{GSp}_4$ explicitly in this section, and consider weights of the form $\kappa=(k_1,k_2,-(k_1+k_2)$ such that the automorphic vector bundle

$$\mathcal{V}_{(k_1,k_2,-(k_1+k_2)}=\mathrm{Sym}^{k_1-k_2}\omega_A\otimes\mathrm{Det}^{k_2}\omega_A,$$

where A is the universal semi-abelian scheme. There are two different ways to discuss slopes of overconvergent cohomology groups (is it obvious that they are equivalent?)

- Take the limit over all levels at p (this doesn't change our overconvergent cohomology groups, as we have seen). Then we have an action of $T^+(\mathbb{Q}_p)$ on our cohomology groups, and there is a way to discuss slopes in this generality (see Section 5.9 of [1]). Basically the cohomology complexes $R\Gamma_w(\kappa)$ will have a collection of slopes λ that live in $X^*(T)_{\mathbb{R}}$. This means that we can compare then to other characters using the Bruhat order.

- Work at Iwahori level, and look at p-adic valuations of eigenvalues of the U_p operators U_1 and U_2. This is much simpler and is what we will do, however we will phrase our results in the abstract formalism of slope characters.

Conjecture 3.11.1 (Conjecture 5.29 of [1]). *Fix $w \in {}^M W$ and $\kappa \in X^*(T)^{M,+}$. Then the slopes of $R\Gamma_w(\kappa)$ are bounded below by*

$$w^{-1}w_{0,M}(\kappa+\rho)+\rho.$$

When $G = \mathrm{GSp}_4$ and $K_p = \mathcal{I}$ (which we can take because the finite slope cohomology groups don't depend on the level) this has the following explicit meaning: If $\kappa = (k_1, k_2; -k_1 - k_2)$ then we expect the the p-adic valuations of the eigenvalues of U_1 and U_2 to be bounded below by the numbers given in the following table (which I took straight from [1]): Note that $w_{0,M} = s_0$

	1	s_1	$s_1 s_0$	$s_1 s_0 s_1$
U_2	3	k_2+1	k_2+1	k_1+k_2
U_1	k_2+3	k_2+3	$2k_2+k_1$	$2k_2+k_1$

Theorem 3.11.2 (Theorem 5.33 of [1]). *Fix $w \in {}^M W$ and $\kappa \in X^*(T)^{M,+}$. Then the slopes of $R\Gamma_w(\kappa)$ are bounded below by*

$$w^{-1}w_{0,M}(\kappa)$$

When $G = \mathrm{GSp}_4$ and $K_p = \mathcal{I}$ we get the following table

	1	s_1	$s_1 s_0$	$s_1 s_0 s_1$
U_2	0	k_2	k_2	k_1+k_2
U_1	k_2	k_2	$2k_2+k_1$	$2k_2+k_1$

Proof of Theorem 3.11.2. The proof doesn't seem to be too involved, they just compute what happens in local coordinates (these computations take up a few pages at the end of Section 4 of [1]). ☐

Remark 3.11.3. The difference between the Theorem and the conjecture is the constant $w^{-1}w_{0,M}\rho+\rho$. Boxer and Pilloni suggest that this term should be related to the geometry of integral models of the relevant Hecke correspondences. As far as I'm aware, there aren't any general constructions in the literature for integral models of such Hecke correspondences with good properties. To be precise, one would like integral models

$$\begin{tikzcd} & \mathscr{C} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathscr{S} & & \mathscr{S} \end{tikzcd}$$

together with a map $p_2^*\mathcal{V}_\kappa \to p_1^!\mathcal{V}_\kappa$, or perhaps even a map $p_2^*\mathcal{V}_\kappa \to p^N p_1^!\mathcal{V}_\kappa$ for some optimal power p^N of p. It should be possible to construct such cohomological correspondences when C is a Shimura variety of parahoric level, see for example [3] for some PEL cases. The results of loc. cit. make crucial use of the fact that the integral models of Shimura varieties of parahoric level are Cohen-Macaulay and flat. It is probably very difficult to construct good integral models with cohomological correspondences in general.

3.12. **Small slopes.** For a given weight κ, we define a small slope condition $ss(\kappa)$ and a strictly small slope condition $sss(\kappa)$ so that the following results hold per definition:

- If Conjecture 5.29 holds, then
$$R\Gamma_w(\kappa)^{ss(\kappa)}$$
is nonzero only for $w \in C(\kappa)^+$.
- The cohomology complexes
$$R\Gamma_w(\kappa)^{sss(\kappa)}$$
are nonzero only for $w \in C(\kappa)^+$.

We recall that
$$C(\kappa)^+ = \{w \in W \mid w^{-1}w_{0,M}(\kappa+\rho) \in X^*(T)^-_{\mathbb{Q}}\}.$$
In particular if $\kappa+\rho$ is regular then the Cousin complex is concentrated in a single column, and by the duality argument which I haven't explained, it follows that it degenerates at E_1 giving a quasi-isomorphism
$$R\Gamma_w(\kappa)^{sss(\kappa)} \simeq R\Gamma(\kappa)^{sss(\kappa)}.$$
Moreover, the cohomology of the above complex is concentrated in degree $l(w)$ for the unique $w \in C(\kappa)^+$.

When $G = \mathrm{GSp}_4$, we expect the following $\kappa = (k_1, k_2; -k_1-k_2)$ to have nonzero cohomology in the following degrees This defines four regions in the plane, which we've drawn them in Figure 3.

H^0	$0 \leq k_2 - 2 \leq k_1 - 1$
H^1	$0 \leq 2 - k_2 \leq k_1 - 1$
H^2	$0 \leq k_1 - 1 \leq 2 - k_2$
H^3	$0 \leq 1 - k_1 \leq 2 - k_2$

References

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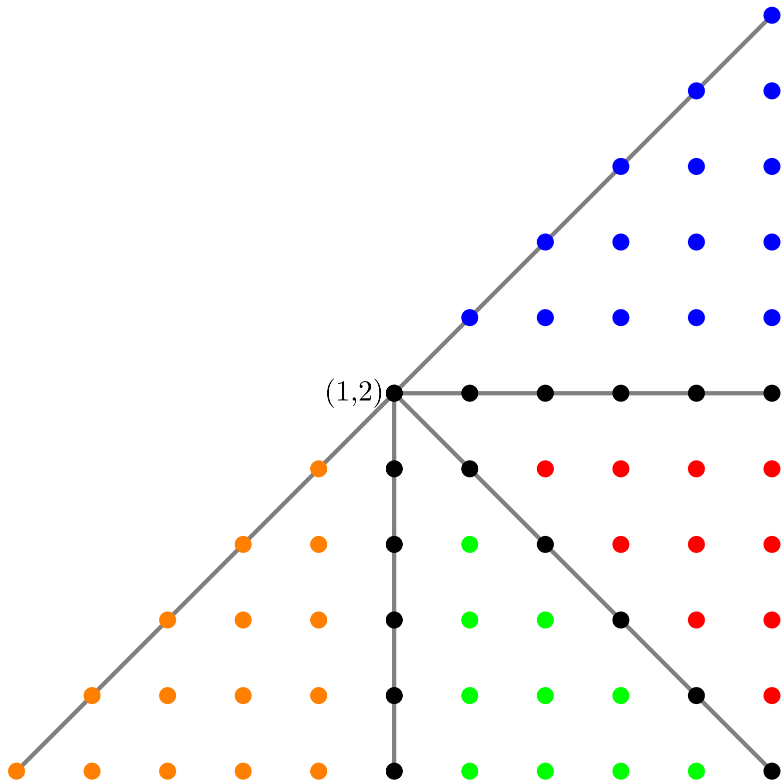

Figure 3. *Weights of automorphic vector bundles for Siegel threefolds are divided into four regions: The coloured dots represent the lattice point that lie in only one of the four regions, with blue corresponding to the weights where we only expect finite slope cohomology in* H^0*, red to* H^1*, green to* H^2 *and orange to* H^3*. The black dots bordering the regions will lie in multiple regions; these are the irregular weights where we expect to see finite slope cohomology in multiple degrees.*